

# THE ISOMORPHISM PROBLEM FOR UNIVERSAL ENVELOPING ALGEBRAS OF NILPOTENT LIE ALGEBRAS

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**ABSTRACT.** In this paper we study the isomorphism problem for the universal enveloping algebras of nilpotent Lie algebras. We prove that if the characteristic of the underlying field is not 2 or 3, then the isomorphism type of a nilpotent Lie algebra of dimension at most 6 is determined by the isomorphism type of its universal enveloping algebra. Examples show that the restriction on the characteristic is necessary.

## 1. INTRODUCTION

In this paper we examine the isomorphism problem for universal enveloping algebras of Lie algebras. It is known that two non-isomorphic Lie algebras may have isomorphic universal enveloping algebras; see for instance [RU, Example A]. However, all such known examples require that the characteristic of the underlying field is a prime. In this paper we focus on nilpotent Lie algebras and prove the following main result.

**Theorem 1.1.** *Suppose that  $\mathbb{F}$  is a field such that  $\text{char } \mathbb{F} \notin \{2, 3\}$ . Then the isomorphism type of a nilpotent Lie algebra of dimension at most 6 over  $\mathbb{F}$  is determined by the isomorphism type of its universal enveloping algebra.*

Theorem 1.1 is a consequence of Theorems 3.1 and 4.1. As shown by examples in Sections 3 and 4, the requirement about the characteristic of the underlying field is necessary. In addition to proving Theorem 1.1, we classify the possible isomorphisms between the universal enveloping algebras of nilpotent Lie algebras of dimension at most 5 over an arbitrary field, and those of dimension 6 over fields of characteristic different from 2 (see Theorems 3.1 and 4.1).

Little progress has ever been made on the isomorphism problem for universal enveloping algebras. For a Lie algebra  $L$ , let  $U(L)$  denote its universal enveloping algebra (see Section 2 for the definitions). Several invariants of  $L$  are known to be determined by  $U(L)$ . For instance, if  $L$  is finite-dimensional, then the (linear) dimension of  $L$  coincides with the Gelfand-Kirillov dimension of  $U(L)$  (see [KL]). More recently Riley and Usefi [RU] proved that the nilpotence of  $L$  is determined by  $U(L)$  and, for a nilpotent  $L$ , the nilpotency class of  $L$  can be computed using  $U(L)$ . Moreover the isomorphism type of  $U(L)$  determines the isomorphism type of the graded algebra  $\text{Gr}(L)$  associated with the lower central series of  $L$  (see Section 2 for the definitions). Malcolmson [M] showed

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that if  $L$  is a 3-dimensional simple Lie algebra over a field of characteristic not 2, then  $L$  is determined by  $U(L)$  up to isomorphism. Later Chun, Kajiwara, and Lee [CKL] generalized Malcolmson's result to the class of all Lie algebras with dimension 3 over fields of characteristic not 2.

By proving Theorem 1.1, we verify that the isomorphism problem for universal enveloping algebras has a positive solution in the class of nilpotent Lie algebras with dimension at most 6 over fields of characteristic different from 2 and 3. The proof of this result relies on the classification of nilpotent Lie algebras with dimension at most 6. The classification of such Lie algebras of dimension at most 5 has been known for a long time over an arbitrary field. In dimension 6, several classifications have been published, but they were often incorrect, and they usually only treated fields of characteristic 0. Recently De Graaf [dG] published a classification of 6-dimensional nilpotent Lie algebras over an arbitrary field of characteristic not 2. As the classification by De Graaf has been obtained making heavy use of computer calculations, and was checked by computer for small fields [Sch], we consider this classification as the most reliable in the literature. The reason we do not treat 6-dimensional nilpotent Lie algebras over fields of characteristic 2 is that, in this case, we do not know of a similarly reliable classification.

Our strategy in proving Theorem 1.1 is to determine all pairs of nilpotent Lie algebras  $L_1, L_2$  with dimension at most 6, such that the graded algebras  $\text{Gr}(L_1)$  and  $\text{Gr}(L_2)$  associated with the lower central series are isomorphic. We know from [RU] that this is a necessary condition for the isomorphism  $U(L_1) \cong U(L_2)$ . Such pairs can be read off from the list of nilpotent Lie algebras with dimension at most 6 in [dG]. Next, for all such pairs, we either argue that  $U(L_1)$  cannot be isomorphic to  $U(L_2)$ , or we exhibit an explicit isomorphism between  $U(L_1)$  and  $U(L_2)$ . Initially, computer experiments played a role in determining the isomorphisms between universal enveloping algebras of nilpotent Lie algebras. Recent work by Eick [E] describes a practical algorithm to decide isomorphism between finite dimensional nilpotent associative algebras. Her algorithm was implemented in the **ModIsom** package [MI] of the **GAP** computational algebra system [GAP] and the implementation works for algebras with dimensions more than 100 over small finite fields. We used this implementation to decide isomorphisms between the finite-dimensional, nilpotent quotients  $\Omega(L)/\Omega^k(L)$  of the augmentation ideals  $\Omega(L)$  of  $U(L)$  (see Section 2 for notation). We remark here an interesting observation. If  $L$  is nilpotent of class  $c$  then based on our calculations the isomorphism type of the quotient  $\Omega(L)/\Omega^{c+1}(L)$  determines the isomorphism type of  $L$ . So, the question remains whether  $\Omega(L)/\Omega^{c+1}(L)$  determines the isomorphism type of  $L$  in all dimensions.

## 2. PRELIMINARIES

In this section we summarize some important facts about universal enveloping algebras of Lie algebras; see [D] for a more detailed background. We assume from now on that Lie algebras are finite-dimensional, even though most of the results referred to in this section hold for a larger class of Lie algebras.

Let  $L$  be a Lie algebra over a field  $\mathbb{F}$ . The universal enveloping algebra  $U(L)$  of  $L$  is defined as follows. For  $i \geq 0$ , let  $L^{\otimes i}$  denote the  $i$ -fold tensor power of  $L$  and set

$$\mathcal{T} = \bigoplus_{i=0}^{\infty} L^{\otimes i}.$$

The space  $L^{\otimes 0}$  is one-dimensional, generated by the unit of  $\mathcal{T}$ , and is usually identified with  $\mathbb{F}$ . The sum  $\mathcal{T}$  can be considered as an algebra over  $\mathbb{F}$  with respect to the obvious multiplication defined on the generators of  $\mathcal{T}$  as

$$(v_1 \otimes \cdots \otimes v_k) \cdot (u_1 \otimes \cdots \otimes u_l) = v_1 \otimes \cdots \otimes v_k \otimes u_1 \otimes \cdots \otimes u_l,$$

for all  $v_1, \dots, v_k, u_1, \dots, u_l \in L$ . The algebra  $\mathcal{T}$  is usually referred to as the *tensor algebra* of  $L$ . Let  $\mathcal{I}$  denote the two-sided ideal of  $\mathcal{T}$  generated by elements of the form  $[u, v] - u \otimes v + v \otimes u$ . Then the *universal enveloping algebra*  $U(L)$  of  $L$  is defined as the quotient  $\mathcal{T}/\mathcal{I}$ . We view  $L = L^{\otimes 1}$  as a Lie subalgebra of  $U(L)$ . Universal enveloping algebras have the following universal property.

**Lemma 2.1.** *Suppose that  $L$  is a Lie algebra,  $A$  is an associative algebra, and let  $\varphi : L \rightarrow A$  be a Lie algebra homomorphism. Then there is a unique associative algebra homomorphism  $\bar{\varphi} : U(L) \rightarrow A$  such that  $\bar{\varphi}|_L = \varphi$ .*

The linear subspace spanned by a subset  $X$  of a vector space is denoted by  $\langle X \rangle_{\mathbb{F}}$ . Recall that the *center*  $Z(L)$  of a Lie algebra  $L$  is the ideal

$$\langle x \in L \mid [x, a] = 0, \text{ for all } a \in L \rangle_{\mathbb{F}}.$$

The center of the universal enveloping algebra plays an important role in our arguments. The *center*  $Z(A)$  of an associative algebra  $A$  is defined as

$$Z(A) = \langle x \in A \mid ax = xa, \text{ for all } a \in A \rangle_{\mathbb{F}}.$$

It is clear that the subalgebra of  $U(L)$  generated by the center  $Z(L)$  of  $L$  lies in  $Z(U(L))$ .

Note that  $\mathcal{T}_+ = L^{\otimes 1} \oplus L^{\otimes 2} \oplus \cdots$  is a two-sided ideal in  $\mathcal{T}$ , and the image of  $\mathcal{T}_+$  in  $U(L)$  is referred to as the *augmentation ideal* of  $U(L)$  and is denoted by  $\Omega(L)$ . The following is proved in [RU, Lemma 2.1].

**Lemma 2.2.** *For Lie algebras  $L$  and  $K$ ,  $U(L) \cong U(K)$  if and only if  $\Omega(L) \cong \Omega(K)$ .*

Investigating the isomorphism between  $U(L)$  and  $U(K)$  will often be carried out, using Lemma 2.2, through studying the isomorphism between  $\Omega(K)$  and  $\Omega(L)$ . For  $i \geq 1$ , let  $\Omega^i(L)$  denote the ideal of  $\Omega(L)$  generated by the products of elements of  $\Omega(L)$  with length at least  $i$ . This way we obtain a descending series in  $\Omega(L)$ :

$$\Omega(L) \supseteq \Omega^2(L) \supseteq \cdots.$$

The sequence  $\Omega^i(L)$  is a filtration on  $\Omega(L)$ : for  $x \in \Omega^i(L)$  and  $y \in \Omega^j(L)$  we have that  $xy \in \Omega^{i+j}(L)$ .

A basis for  $\Omega^i(L)$  can usually be constructed as follows. Let  $L^i$  denote the  $i$ -th term of the *lower central series* of  $L$ ; that is  $L^1 = L$ , and, for  $i \geq 1$ ,  $L^{i+1} = [L^i, L]$ . For

an element  $v \in L$ , we define the *weight*  $w(v)$  of  $v$  as the largest integer  $i$  such that  $v \in L^i$ . A basis  $\mathcal{B} = \{v_1, \dots, v_d\}$  of a Lie algebra  $L$  is said to be *homogeneous* if the basis elements with weight at least  $i$  form a basis for  $L^i$ . An element of  $U(L)$  of the form  $m = v_{i_1}v_{i_2}\dots v_{i_k}$  with  $i_1 \leq i_2 \leq \dots \leq i_k$  is said to be a *Poincaré-Birkhoff-Witt monomial*, or more briefly, a *PBW monomial*, in  $\mathcal{B}$ . The *weight*  $w(m)$  of such a monomial is defined as  $w(v_{i_1}) + w(v_{i_2}) + \dots + w(v_{i_k})$ .

**Theorem 2.3** (Proposition 3.1(1) in [R]). *Let  $L$  be a Lie algebra with a homogeneous basis  $\mathcal{B}$  and let  $t \geq 1$ . Then the set of all PBW monomials in  $\mathcal{B}$  with weight at least  $t$  forms an  $\mathbb{F}$ -basis for  $\Omega^t(L)$ , for every  $t \geq 1$ .*

In this paper we are interested to discover, for a pair of nilpotent non-isomorphic Lie algebras  $L_1$  and  $L_2$ , if  $U(L_1)$  can be isomorphic to  $U(L_2)$ . Often we use the graded algebras  $\text{Gr}(L_i)$  associated with the lower central series of the  $L_i$  to rule out the isomorphism between  $U(L_1)$  and  $U(L_2)$ . For a Lie algebra  $L$ ,  $\text{Gr}(L)$  is defined as the algebra on the linear space

$$\text{Gr}(L) = L/L^2 \oplus L^2/L^3 \oplus \dots$$

with respect to the multiplication given by the rule

$$[x + L^{i+1}, y + L^{j+1}] = [x, y] + L^{i+j+1} \quad \text{for all } x \in L^i \text{ and } y \in L^j.$$

The following is proved in [RU, Proposition 4.1].

**Theorem 2.4.** *For any Lie algebra  $L$ , the isomorphism type of  $U(L)$  determines the isomorphism type of  $\text{Gr}(L)$ . Consequently, if  $L$  is nilpotent of class 2, then the isomorphism type of  $U(L)$  determines the isomorphism type of  $L$ .*

Suppose that  $L_1$  and  $L_2$  are nilpotent Lie algebras such that  $U(L_1) \cong U(L_2)$ . Then Theorem 2.4 implies, for all  $i$ , that  $\dim L_1^i = \dim L_2^i$ , and that the nilpotency classes of  $L_1$  and  $L_2$  coincide. The second statement of Theorem 2.4 is an easy consequence of the first statement if  $L$  is finite-dimensional. Indeed, if  $L$  is a finite-dimensional Lie algebra of nilpotency class 2, then it is always isomorphic to  $\text{Gr}(L)$ . The same assertion without a restriction on the dimension is proved in [RU, Section 5].

The following lemma can be found in [RU, Lemma 5.1].

**Lemma 2.5.** *Let  $L$  and  $K$  be Lie algebras and let  $\varphi : \Omega(L) \rightarrow \Omega(K)$  be an isomorphism. Then  $\varphi(L^i + \Omega^{i+1}(L)) = K^i + \Omega^{i+1}(K)$ , for every positive integer  $i$ .*

### 3. NILPOTENT LIE ALGEBRAS WITH DIMENSION AT MOST 5

In this section we determine all isomorphisms between the universal enveloping algebras of nilpotent Lie algebras with dimension at most 5. A classification of such Lie algebras is well known and can be found, for instance, in [dG]. The main result of this section is the following.

**Theorem 3.1.** *Let  $L$  and  $K$  be nilpotent Lie algebras of dimension at most 5 over a field  $\mathbb{F}$  such that  $U(L) \cong U(K)$ . Then one of the following must hold:*

- (i)  $L \cong K$ ;
- (ii)  $\text{char } \mathbb{F} = 2$ ; further  $L$  and  $K$  are isomorphic to the Lie algebras  $L_{5,3}$  and  $L_{5,5}$  or they are isomorphic to the Lie algebras  $L_{5,6}$  and  $L_{5,7}$  in Section 5 of [dG].

Since there is a unique isomorphism class of nilpotent Lie algebras with dimension 1, and there is a unique such class with dimension 2, the isomorphism problem of universal enveloping algebras is trivial in these cases. Up to isomorphism, there are two nilpotent Lie algebras with dimension 3, an abelian, and a non-abelian. By Theorem 2.4 their universal enveloping algebras must be non-isomorphic. The number of isomorphism classes of 4-dimensional nilpotent Lie algebras is 3. One of these algebras is abelian, the second has nilpotency class 2, and the third has nilpotency class 3. Again, by Theorem 2.4, their universal enveloping algebras are pairwise non-isomorphic. This proves Theorem 3.1 for Lie algebras of dimension at most 4.

There are 9 isomorphism classes of nilpotent Lie algebras with dimension 5 and they are listed at the beginning of Section 4 in [dG]. To simplify notation, we denote the Lie algebra  $L_{5,i}$  in De Graaf's list by  $L_i$ . Inspecting this list, we find that the sequence  $(\dim L^1, \dim L^2, \dots)$  for these Lie algebras, after omitting the trailing zeros, are as follows: (5), (5, 1), (5, 2, 1), (5, 1), (5, 2, 1), (5, 3, 2, 1), (5, 3, 2, 1), (5, 2), (5, 3, 2). Therefore Theorem 2.4 implies that if  $U(L_i) \cong U(L_j)$  then either  $i = j$ , or  $\{i, j\} = \{3, 5\}$ , or  $\{i, j\} = \{6, 7\}$ . The Lie algebras that are involved in these possible isomorphisms are as follows:

$$\begin{aligned}
L_3 &= \langle x_1, x_2, x_3; x_4; x_5 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5 \rangle; \\
L_5 &= \langle x_1, x_2, x_3; x_4; x_5 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle; \\
L_6 &= \langle x_1, x_2, x_3; x_4; x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle; \\
L_7 &= \langle x_1, x_2, x_3; x_4; x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 \rangle.
\end{aligned}$$

Products that are zero are omitted from the multiplication tables above. For instance  $[x_1, x_3] = 0$  in  $L_3$  and in  $L_5$ . Note that the bases of the Lie algebras given above are homogeneous. Further, basis elements of different weights are separated by a semicolon. The multiplication tables given in [dG] are somewhat different from the ones above, as we changed the orders of certain basis elements in order to work with homogeneous bases. A possible source of confusion is that the symbols  $x_i$  are used to denote elements of different Lie algebras, but we believe that using different letters or introducing a subscript or superscript would unnecessarily complicate the notation.

**Lemma 3.2.** *If  $\mathbb{F}$  is a field of characteristic not 2, then  $\Omega(L_3)/\Omega^4(L_3) \not\cong \Omega(L_5)/\Omega^4(L_5)$  and  $\Omega(L_6)/\Omega^5(L_6) \not\cong \Omega(L_7)/\Omega^5(L_7)$ ; consequently  $\Omega(L_3) \not\cong \Omega(L_5)$  and  $\Omega(L_6) \not\cong \Omega(L_7)$ . Otherwise,  $\Omega(L_3) \cong \Omega(L_5)$  and  $\Omega(L_6) \cong \Omega(L_7)$ .*

*Proof.* Suppose that  $\text{char } \mathbb{F} \neq 2$  and set  $B_i = \Omega(L_i)/\Omega^4(L_i)$ , for  $i = 3, 5$ . First we show that  $B_3 \not\cong B_5$ . We claim that  $Z(B_5) \leq (B_5)^2$  while  $Z(B_3) \not\leq (B_3)^2$ , which will imply that  $B_3 \not\cong B_5$ . As  $x_3 \in Z(B_3) \setminus (B_3)^2$ , the second assertion of the claim is valid. In order to prove the first assertion, let  $w \in Z(B_5)$ . We write  $w$  as a linear combination of PBW

monomials:

$$w = \sum_{i_1 \leq \dots \leq i_n} \alpha_{i_1, \dots, i_n} x_{i_1} \cdots x_{i_n}.$$

Then  $w \equiv \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \pmod{(B_5)^2}$ . Since  $w$  is a central element in  $B_5$ , we have  $[x_1, w] = [x_2, w] = 0$ , and so

$$\begin{aligned} 0 &= [x_1, w] \equiv \alpha_2 [x_1, x_2] = \alpha_2 x_4 \pmod{(B_5)^3}, \\ 0 &= [x_2, w] \equiv \alpha_1 [x_2, x_1] = -\alpha_1 x_4 \pmod{(B_5)^3}. \end{aligned}$$

Hence,  $\alpha_1 = \alpha_2 = 0$ . Since  $x_5 \in L_5^3$ , we have

$$\begin{aligned} 0 &= [x_2, w] = \alpha_3 [x_2, x_3] + \alpha_{1,1} [x_2, x_1^2] + \alpha_{1,2} [x_2, x_1 x_2] + \alpha_{1,3} [x_2, x_1 x_3] \\ &= \alpha_3 x_5 + \alpha_{1,1} (-2x_1 x_4 + x_5) - \alpha_{1,2} x_2 x_4 - \alpha_{1,3} x_3 x_4. \end{aligned}$$

Since  $\text{char } \mathbb{F} \neq 2$ , we deduce that  $\alpha_3 = 0$ . Thus,  $w \in (B_5)^2$  as claimed.

Now set  $B_i = \Omega(L_i)/\Omega^5(L_i)$ , for  $i = 6, 7$ . Suppose, to the contrary, that  $f : B_7 \rightarrow B_6$  is an isomorphism. Since  $B_7$  is generated by  $x_1$  and  $x_2$ , the map  $f$  is determined by the images  $f(x_1)$  and  $f(x_2)$ . As above, let us write  $f(x_1)$  and  $f(x_2)$  as linear combinations of PBW monomials:

$$\begin{aligned} f(x_1) &= \sum_{i_1 \leq \dots \leq i_n} \alpha_{i_1, \dots, i_n} x_{i_1} \cdots x_{i_n}, \\ f(x_2) &= \sum_{i_1 \leq \dots \leq i_n} \beta_{i_1, \dots, i_n} x_{i_1} \cdots x_{i_n}. \end{aligned}$$

As

$$(1) \quad f([x_1, x_2]) \equiv (\alpha_1 \beta_2 - \alpha_2 \beta_1) [x_1, x_2] = (\alpha_1 \beta_2 - \alpha_2 \beta_1) x_3 \pmod{(B_6)^3},$$

we obtain that  $\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0$ . Equation (1) also gives

$$0 = f([x_1, x_2, x_2]) \equiv \beta_1 (\alpha_1 \beta_2 - \alpha_2 \beta_1) [x_1, x_2, x_1] = -\beta_1 (\alpha_1 \beta_2 - \alpha_2 \beta_1) x_4 \pmod{(B_6)^4}.$$

Since  $\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0$ , we must have  $\beta_1 = 0$ . This implies that  $\alpha_1 \neq 0$  and  $\beta_2 \neq 0$ . Furthermore, modulo  $(B_6)^4$ , we have

$$\begin{aligned} f([x_1, x_2]) &\equiv \alpha_1 \beta_2 x_3 + (\alpha_1 \beta_3 + \alpha_2 \beta_{1,1} - \alpha_{1,1} \beta_2) x_4 + \\ &\quad (\alpha_1 \beta_{1,2} - 2\alpha_2 \beta_{1,1} + 2\alpha_{1,1} \beta_2) x_1 x_3 + (2\alpha_1 \beta_{2,2} - \alpha_2 \beta_{1,2} + \alpha_{1,2} \beta_2) x_2 x_3. \end{aligned}$$

Hence  $f([x_1, x_2, x_2])$  is equal to the following:

$$(-\alpha_1 \beta_2^2 + \alpha_1 \beta_2 \beta_{1,1}) x_5 - 2\alpha_1 \beta_2 \beta_{1,1} x_1 x_4 - \alpha_1 \beta_2 \beta_{1,2} x_2 x_4 + \beta_2 (\alpha_1 \beta_{1,2} - 2\alpha_2 \beta_{1,1} + 2\alpha_{1,1} \beta_2) x_3 x_3.$$

As  $[x_1, x_2, x_2] = 0$  in  $\Omega(L_7)$ , we must have  $f([x_1, x_2, x_2]) = 0$ . Since  $\text{char } (\mathbb{F}) \neq 2$  and  $\alpha_1 \beta_2 \neq 0$ , we get  $\beta_{1,1} = 0$ . Thus the coefficient of  $x_5$  in  $f([x_1, x_2, x_2])$  is  $-\alpha_1 \beta_2^2$ . This, implies that  $f([x_1, x_2, x_2]) \neq 0$ , which is a contradiction. Thus  $B_6 \not\cong B_7$ , as claimed.

Let us now assume that  $\text{char } \mathbb{F} = 2$ , and prove  $\Omega(L_3) \cong \Omega(L_5)$  and  $\Omega(L_6) \cong \Omega(L_7)$ . Note that the map from  $L_3$  to  $\Omega(L_5)$  induced by  $x_1 \mapsto x_1$ ,  $x_2 \mapsto x_2$ ,  $x_3 \mapsto x_3 + x_1^2$  is a Lie homomorphism. Hence, by Lemma 2.1, it can be extended to a homomorphism  $\varphi$

from  $\Omega(L_3)$  to  $\Omega(L_5)$ . Since  $x_1$ ,  $x_2$ , and  $x_3 + x_1^2$  form a generating set for  $\Omega(L_5)$ ,  $\varphi$  is onto. We need to show that  $\varphi$  is injective. Since  $\dim \Omega(L_3)/\Omega^i(L_3) = \dim \Omega(L_5)/\Omega^i(L_5)$  (see Theorem 2.3), the homomorphism  $\varphi$  induces an isomorphism

$$\varphi_i : \Omega(L_3)/\Omega^i(L_3) \rightarrow \Omega(L_5)/\Omega^i(L_5),$$

for every  $i \geq 1$ . Let  $x$  be a non-zero element in  $\Omega(L_3)$  such that  $\varphi(x) = 0$ . By Theorem 2.3, there exists a positive integer  $i$  such that  $x \in \Omega^{i-1}(L_3) \setminus \Omega^i(L_3)$ . As  $\varphi_i$  is an isomorphism, we get  $\varphi_i(x) \neq 0$ , which is a contradiction. Hence  $\varphi$  is injective, and so  $\Omega(L_3) \cong \Omega(L_5)$ .

One can show precisely the same way that  $x_1 \mapsto x_1$ ,  $x_2 \mapsto x_2 + x_1^2$  extends to an isomorphism from  $\Omega(L_7)$  to  $\Omega(L_6)$ .  $\square$

Lemma 3.2 in combination with Lemma 2.2, and the argument preceding it proves Theorem 3.1.

#### 4. 6-DIMENSIONAL NILPOTENT LIE ALGEBRAS

The isomorphisms between universal enveloping algebras of nilpotent Lie algebras of dimension 6 in characteristic different from 2 are described by the following theorem.

**Theorem 4.1.** *Let  $L$  and  $K$  be nilpotent Lie algebras of dimension 6 over a field  $\mathbb{F}$  of characteristic not 2. If  $U(L) \cong U(K)$ , then one of the following must hold.*

- (i)  $L \cong K$ .
- (ii)  $\text{char } \mathbb{F} = 3$ ; further  $L$  and  $K$  are isomorphic to one of the following pairs of Lie algebras in [dG, Section 5]:  $L_{6,6}$  and  $L_{6,11}$ ;  $L_{6,7}$  and  $L_{6,12}$ ;  $L_{6,17}$  and  $L_{6,18}$ ;  $L_{6,23}$  and  $L_{6,25}$ .

Throughout this section  $\mathbb{F}$  denotes a field with characteristic different from 2. The proof of Theorem 4.1 is presented in this section. De Graaf [dG] lists the isomorphism types of 6-dimensional nilpotent Lie algebras over an arbitrary field  $\mathbb{F}$  of characteristic different from 2. De Graaf denotes these Lie algebras by  $L_{6,i}$  or  $L_{6,i}(\varepsilon)$  where  $1 \leq i \leq 26$  and  $\varepsilon$  is a field element. To simplify notation and to distinguish between the Lie algebras of Sections 3 and 4 we will denote  $L_{6,i}$  with  $K_i$  and  $L_{6,i}(\varepsilon)$  with  $K_i(\varepsilon)$ . Inspecting the list of Lie algebras, we obtain that the isomorphisms among the graded Lie algebras associated with the lower central series of these Lie algebras are as follows:

- (1)  $\text{Gr}(K_3) \cong \text{Gr}(K_5) \cong \text{Gr}(K_{10})$ ;
- (2)  $\text{Gr}(K_6) \cong \text{Gr}(K_7) \cong \text{Gr}(K_{11}) \cong \text{Gr}(K_{12}) \cong \text{Gr}(K_{13})$ ;
- (3)  $\text{Gr}(K_{14}) \cong \text{Gr}(K_{16})$ ;
- (4)  $\text{Gr}(K_{15}) \cong \text{Gr}(K_{17}) \cong \text{Gr}(K_{18})$ ;
- (5)  $\text{Gr}(K_{23}) \cong \text{Gr}(K_{25})$ ;
- (6)  $\text{Gr}(K_{24}(\varepsilon)) \cong \text{Gr}(K_9)$ , for all  $\varepsilon \in \mathbb{F}$ .

Let  $L$  and  $K$  be 6-dimensional non-isomorphic nilpotent Lie algebras over  $\mathbb{F}$ . If  $U(L) \cong U(K)$  then, by Theorem 2.4,  $L$  and  $K$  must both occur in one of the lines (1)–(6) above. In Lemmas 4.2–4.7 we examine the possible isomorphisms between the universal

enveloping algebras of the Lie algebras that occur in one of the families above. Using Lemma 2.2 we only examine the possible isomorphisms between the augmentation ideals.

As with the 5-dimensional Lie algebras, we change the multiplication tables of the algebras presented in [dG] in order to work with homogeneous bases. The bases elements of different weights are separated by a semicolon. Further, as in Section 3, we omit products of the form  $[x_i, x_j] = 0$  from the multiplication tables of the Lie algebras. For  $i = 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 23, 25$  we set  $U_i = U(K_i)$  and  $\Omega_i = \Omega(K_i)$ . Further, let  $U_{24}(\varepsilon)$  and  $\Omega_{24}(\varepsilon)$  denote  $U(K_{24}(\varepsilon))$  and  $\Omega(K_{24}(\varepsilon))$ , respectively.

**4.1. Family (1).** First we deal with the isomorphism  $\text{Gr}(K_3) \cong \text{Gr}(K_5) \cong \text{Gr}(K_{10})$  where

$$\begin{aligned} K_3 &= \langle x_1, x_2, x_3, x_4; x_5; x_6 \mid [x_1, x_2] = x_5, [x_1, x_5] = x_6 \rangle; \\ K_5 &= \langle x_1, x_2, x_3, x_4; x_5; x_6 \mid [x_1, x_2] = x_5, [x_1, x_5] = x_6, [x_2, x_3] = x_6 \rangle; \\ K_{10} &= \langle x_1, x_2, x_3, x_4; x_5; x_6 \mid [x_1, x_2] = x_5, [x_1, x_5] = x_6, [x_3, x_4] = x_6 \rangle. \end{aligned}$$

**Lemma 4.2.** *The algebras  $\Omega_3$ ,  $\Omega_5$ , and  $\Omega_{10}$  are pairwise non-isomorphic.*

*Proof.* For  $i = 3, 5, 10$ , let  $B_i = \Omega_i/(\Omega_i)^4$ . It is enough to prove that the centers  $Z_i = Z(B_i)$  have different dimensions. The quotient  $B_i$  is spanned by the images of the PBW monomials with weight at most 3 and we will identify such a monomial with its image. We claim that  $Z_3 = \langle x_3, x_4, (B_3)^3 \rangle_{\mathbb{F}}$ . Set  $C = \langle x_3, x_4, (B_3)^3 \rangle_{\mathbb{F}}$ . Clearly,  $C \leq Z_3$ . Let  $z \in Z_3$ . Then  $z$  is a linear combination of PBW monomials with weight at most 3. As usual,  $\alpha_{i_1, \dots, i_n}$  is the coefficient of  $x_{i_1} \cdots x_{i_n}$  in the PBW representation of  $z$ . We may assume without loss of generality that all monomials in  $C$  occur with coefficient zero. First we compute that

$$[x_1, z] = \alpha_2 x_5 + \alpha_{1,2} x_1 x_5 + 2\alpha_{2,2} x_2 x_5 + \alpha_{2,3} x_3 x_5 + \alpha_{2,4} x_4 x_5 + \alpha_5 x_6.$$

We deduce that  $\alpha_2 = \alpha_{1,2} = \alpha_{2,2} = \alpha_{2,3} = \alpha_{2,4} = \alpha_5 = 0$ . Now

$$[z, x_2] = \alpha_1 x_5 + 2\alpha_{1,1} x_1 x_5 - \alpha_{1,1} x_6 + \alpha_{1,3} x_3 x_5 + \alpha_{1,4} x_4 x_5.$$

This implies that  $\alpha_1 = \alpha_{1,1} = \alpha_{1,3} = \alpha_{1,4} = 0$ . Therefore  $Z_3 = C$  as claimed. Since  $x_3, x_4, x_3 x_3, x_3 x_4, x_4 x_4$  together with the PBW monomials with weight 3 form a basis for  $C$ , we obtain that  $\dim C = \dim Z_3 = 30$ . Similar argument shows that  $\dim Z_5 = 29$ , and  $\dim Z_{10} = 28$ . Thus the  $Z_i$  have different dimensions, as required.  $\square$

**4.2. Family (2).** We examine the following family of Lie algebras.



$$\begin{aligned}
K_6 &= \langle x_1, x_2, x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6 \rangle; \\
K_7 &= \langle x_1, x_2, x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6 \rangle; \\
K_{11} &= \langle x_1, x_2, x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6, \\
&\quad [x_2, x_3] = x_6 \rangle; \\
K_{12} &= \langle x_1, x_2, x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, [x_2, x_3] = x_6 \rangle; \\
K_{13} &= \langle x_1, x_2, x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, [x_2, x_3] = x_5, \\
&\quad [x_4, x_3] = x_6 \rangle.
\end{aligned}$$

**Lemma 4.3.** *The enveloping algebras  $\Omega_6$ ,  $\Omega_7$ ,  $\Omega_{11}$ ,  $\Omega_{12}$ , and  $\Omega_{13}$  are pairwise non-isomorphic provided that  $\text{char } \mathbb{F} \neq 3$ . If  $\text{char } \mathbb{F} = 3$  then  $\Omega_6 \cong \Omega_{11}$  and  $\Omega_7 \cong \Omega_{12}$  and there is no more isomorphism among the algebras in this family.*

*Proof.* We claim, for  $i \in \{6, 7, 11, 12\}$ , that  $\Omega_{13} \not\cong \Omega_i$ . Note that  $L_3 \cong K_i/K_i^4$ , while  $L_5 \cong K_{13}/K_{13}^4$ , where  $L_3$  and  $L_5$  are 5-dimensional Lie algebras defined in Section 3. Thus,  $\Omega(L_3)/\Omega^5(L_3) \cong \Omega_i/(K_i^4 + \Omega_i^5)$  and  $\Omega(L_5)/\Omega^5(L_5) \cong \Omega_{13}/(K_{13}^4 + \Omega_{13}^5)$ . Suppose that  $f : \Omega_i \rightarrow \Omega_{13}$  is an isomorphism. By Lemma 2.5,  $f(K_i^4 + \Omega_i^5) = K_{13}^4 + \Omega_{13}^5$ . Hence,  $f$  induces an isomorphism between  $\Omega_i/(K_i^4 + \Omega_i^5)$  and  $\Omega_{13}/(K_{13}^4 + \Omega_{13}^5)$ . Thus,

$$\Omega(L_3)/\Omega^5(L_3) \cong \Omega(L_5)/\Omega^5(L_5).$$

However Lemma 3.2 shows that  $\Omega(L_3)/\Omega^4(L_3) \not\cong \Omega(L_5)/\Omega^4(L_5)$ . This contradiction implies that  $\Omega_{13} \not\cong \Omega_i$ , as claimed.

Next we show that  $\Omega_6 \not\cong \Omega_7$  and that  $\Omega_{11} \not\cong \Omega_{12}$ . First we argue that  $\Omega_6 \not\cong \Omega_7$ . Suppose on the contrary that  $f : \Omega_7 \rightarrow \Omega_6$  is an isomorphism. Then  $f$  is determined by the images  $f(x_1)$ ,  $f(x_2)$ ,  $f(x_3)$ . Write  $f(x_i)$  as a linear combination of PBW monomials and let  $\alpha_{i_1, \dots, i_n}$ ,  $\beta_{i_1, \dots, i_n}$  and  $\gamma_{i_1, \dots, i_n}$  denote the coefficients of  $x_{i_1} \cdots x_{i_n}$  in  $f(x_1)$ ,  $f(x_2)$ ,  $f(x_3)$ , respectively. Since  $x_3 \in Z(\Omega_7)$ , we have

$$0 = [x_1, f(x_3)] \equiv \gamma_2 x_4 \pmod{(\Omega_6)^3} \quad \text{and} \quad 0 = [x_2, f(x_3)] \equiv -\gamma_1 x_4 \pmod{(\Omega_6)^3}.$$

Therefore  $\gamma_1 = \gamma_2 = 0$ . Let us compute modulo  $(\Omega_6)^4$  that

$$0 = f([x_2, x_1, x_2]) \equiv \beta_1(\alpha_1 \beta_2 - \beta_1 \alpha_2) x_5.$$

If  $\alpha_1 \beta_2 - \beta_1 \alpha_2 = 0$ , then, as  $\gamma_1 = \gamma_2 = 0$ , it follows that  $f(x_i)$  are linearly dependent modulo  $(\Omega_6)^2$ , which is impossible. Thus  $\beta_1 = 0$  and  $\alpha_1 \beta_2 \neq 0$ . Now computation shows, modulo  $(\Omega_6)^5$ , that

$$\begin{aligned}
0 = f([x_1, x_2, x_2]) \equiv & \\
& (-\alpha_1 \beta_2^2 + \alpha_1 \beta_2 \beta_{11}) x_6 - 2\alpha_1 \beta_2 \beta_{1,1} x_1 x_5 - \alpha_1 \beta_2 \beta_{1,2} x_2 x_5 - \alpha_1 \beta_2 \beta_{1,3} x_3 x_5 + \delta \beta_2 x_4 x_4
\end{aligned}$$

where  $\delta = \alpha_1 \beta_{1,2} - 2\alpha_2 \beta_{1,1} + 2\alpha_{1,1} \beta_2$ . Considering the coefficients of  $x_6$  and  $x_1 x_5$ , we deduce that either  $\alpha_1 = 0$  or  $\beta_2 = 0$ , which is a contradiction.

Let us now show that  $\Omega_{11} \not\cong \Omega_{12}$ . Assume by contradiction that  $f : \Omega_{12} \rightarrow \Omega_{11}$  is an isomorphism. Then  $f$  is determined by the images  $f(x_1)$ ,  $f(x_2)$ , and  $f(x_3)$ . As above,

we write  $f(x_i)$  as a linear combination of PBW monomials and we let  $\alpha_{i_1, \dots, i_n}$ ,  $\beta_{i_1, \dots, i_n}$  and  $\gamma_{i_1, \dots, i_n}$  denote the coefficients of  $x_{i_1} \cdots x_{i_n}$  in  $f(x_1)$ ,  $f(x_2)$ ,  $f(x_3)$ , respectively. Let us compute modulo  $(\Omega_{11})^3$  that

$$0 = f([x_1, x_3]) \equiv (\alpha_1 \gamma_2 - \gamma_1 \alpha_2) x_4$$

and

$$0 \equiv f([x_2, x_3]) \equiv (\beta_1 \gamma_2 - \beta_2 \gamma_1) x_4.$$

If the vector  $(\gamma_1, \gamma_2)$  is not zero, then the vectors  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$  must be its scalar multiples and so the  $f(x_i)$  are linearly dependent modulo  $(\Omega_{11})^2$ . Hence  $\gamma_1 = \gamma_2 = 0$ . Now we get a contradiction using the same argument as in the previous paragraph.

Now assume that  $\text{char } \mathbb{F} \neq 3$ . Set  $B_i = \Omega_i / (\Omega_i)^5$ , for  $i = 6, 7, 11, 12$ . Let  $Z_i$  denote the center of the  $B_i$ . We claim that  $\dim Z_6 = \dim Z_7 = 29$  and  $\dim Z_{11} = \dim Z_{12} = 28$ . We only compute  $\dim Z_7$ , as the computation of the other  $Z_i$  is very similar. We claim that

$$Z_7 = \langle x_3, x_3 x_3, x_3 x_3 x_3, (B_7)^4 \rangle_{\mathbb{F}}.$$

Set  $C = \langle x_3, x_3 x_3, x_3 x_3 x_3, (B_7)^4 \rangle_{\mathbb{F}}$ . Clearly,  $C \leq Z_7$ . Let  $z \in Z_7$  and write  $z$  as a linear combination of PBW monomials with weight at most 4. We may assume that all PBW monomials in  $C$  occur with coefficient 0 in  $z$ . Let  $\beta_{i_1, \dots, i_k}$  be the coefficient of  $x_{i_1} \cdots x_{i_k}$ . Then  $\beta_3 = \beta_{3,3} = \beta_{3,3,3} = 0$ . First we obtain that

$$0 = [z, x_5] = \beta_1 [x_1, x_5] = \beta_1 x_6,$$

and so  $\beta_1 = 0$ . Also,  $[z, x_1^3] = -3\beta_2 x_1^2 x_4 + 3\beta_2 x_1 x_5 - \beta_2 x_6$ . So,  $\beta_2 = 0$ . Next compute that

$$\begin{aligned} [z, x_1^2] &= -2\beta_{1,2} x_1^2 x_4 - 4\beta_{2,2} x_1 x_2 x_4 - 2\beta_{2,3} x_1 x_3 x_4 \\ &\quad + (\beta_{1,2} - 2\beta_4) x_1 x_5 + 2\beta_{2,2} x_2 x_5 + \beta_{2,3} x_3 x_5 + 2\beta_{2,2} x_4 x_4 + \beta_4 x_6. \end{aligned}$$

This implies that  $\beta_{1,2} = \beta_{2,2} = \beta_{2,3} = \beta_4 = 0$ . Further,

$$[z, x_2^2] = 4\beta_{1,1} x_1 x_2 x_4 - 2\beta_{1,1} x_2 x_5 - 2\beta_{1,1} x_4 x_4 + 2\beta_{1,3} x_2 x_3 x_4.$$

Thus  $\beta_{1,1} = \beta_{1,3} = 0$ . Then

$$\begin{aligned} 0 = [z, x_1] &= -\beta_{1,1,2} x_1 x_1 x_4 - 2\beta_{1,2,2} x_1 x_2 x_4 - \beta_{1,2,3} x_1 x_3 x_4 - 3\beta_{2,2,2} x_2 x_2 x_4 - 2\beta_{2,2,3} x_2 x_3 x_4 \\ &\quad - \beta_{2,3,3} x_3 x_3 x_4 - \beta_{1,4} x_1 x_5 - \beta_{2,4} x_4^2 - \beta_{2,4} x_2 x_5 - \beta_{3,4} x_3 x_5 - \beta_5 x_6, \end{aligned}$$

and so  $\beta_{1,1,2} = \beta_{1,2,2} = \beta_{1,2,3} = \beta_{2,2,2} = \beta_{2,2,3} = \beta_{2,3,3} = \beta_{1,4} = \beta_{2,4} = \beta_{3,4} = \beta_5 = 0$ . Further

$$\begin{aligned} 0 = [z, x_2] &= 3\beta_{1,1,1} x_1 x_1 x_4 - 3\beta_{1,1,1} x_1 x_5 + \beta_{1,1,1} x_6 \\ &\quad + 2\beta_{1,1,3} x_1 x_3 x_4 - \beta_{1,1,3} x_3 x_5 + \beta_{1,3,3} x_3 x_3 x_4. \end{aligned}$$

Hence  $\beta_{1,1,1} = \beta_{1,1,3} = \beta_{1,3,3} = 0$ . Thus  $z \in C$ , and so  $C = Z_7$  as required. Similar calculations show that

$$\begin{aligned} Z_6 &= \langle x_3, x_3x_3, x_3x_3x_3, (B_6)^4 \rangle_{\mathbb{F}}; \\ Z_{11} &= \langle x_3x_3, x_3x_3x_3, (B_6)^4 \rangle_{\mathbb{F}}; \\ Z_{12} &= \langle x_3x_3, x_3x_3x_3, (B_6)^4 \rangle_{\mathbb{F}}. \end{aligned}$$

Thus the dimensions of  $Z_i$  are as claimed. Hence, if  $\text{char } \mathbb{F} \neq 3$  then,  $\Omega_6 \not\cong \Omega_{11}$ ,  $\Omega_{12}$  and  $\Omega_7 \not\cong \Omega_{11}$ ,  $\Omega_{12}$ .

Combining the results of the last two paragraph, we obtain that the algebras  $\Omega_6$ ,  $\Omega_7$ ,  $\Omega_{11}$ ,  $\Omega_{12}$ ,  $\Omega_{13}$  are pairwise non-isomorphic if  $\text{char } \mathbb{F} \neq 3$ .

Now suppose that  $\text{char } \mathbb{F} = 3$ . We are required to show that  $\Omega_6 \not\cong \Omega_{12}$ ,  $\Omega_7 \not\cong \Omega_{11}$ ,  $\Omega_6 \cong \Omega_{11}$  and that  $\Omega_7 \cong \Omega_{12}$ . Suppose that  $f : \Omega_{12} \rightarrow \Omega_6$  is an isomorphism. Write  $f(x_2)$  and  $f(x_4)$  as linear combinations of PBW monomials and let  $\beta_{i_1, \dots, i_n}$  and  $\gamma_{i_1, \dots, i_n}$  denote the coefficient of  $x_{i_1} \cdots x_{i_n}$  in  $f(x_2)$  and  $f(x_4)$  respectively. By Lemma 2.5,  $f(x_4) \equiv \gamma_4 x_4 \pmod{(\Omega_6)^3}$ . Then, modulo  $(\Omega_6)^4$ ,  $0 = f([x_2, x_4]) \equiv \beta_1 \gamma_4 x_5$ , which gives that  $\beta_1 = 0$ . As  $x_4 = [x_1, x_2]$ , we find that

$$\gamma_{1,1,1} = \gamma_{1,1,2} = \gamma_{1,1,3} = \gamma_{1,2,2} = \gamma_{1,2,3} = \gamma_{1,3,3} = \gamma_{2,2,2} = \gamma_{2,2,3} = \gamma_{2,3,3} = \gamma_{3,3,3} = 0.$$

Then

$$0 = f([x_2, x_4]) = (\beta_2 \gamma_4 - \beta_{1,1} \gamma_4) x_6 - \beta_2 \gamma_{1,4} x_4 x_4 + \beta_{1,2} \gamma_4 x_2 x_5 + \beta_{1,3} \gamma_4 x_3 x_5 - \beta_{1,1} \gamma_4 x_1 x_5.$$

This implies that  $\beta_2 \gamma_4 = 0$ , and in turn that  $\beta_2 = 0$ . However, this gives that  $f(x_1)$ ,  $f(x_2)$ ,  $f(x_3)$  are linearly dependent modulo  $(\Omega_6)^2$ , which is impossible. Hence  $\Omega_6 \not\cong \Omega_{12}$ , and very similar argument shows that  $\Omega_7 \not\cong \Omega_{11}$ .

Finally we need to show that if  $\text{char } \mathbb{F} = 3$ , then  $\Omega_6 \cong \Omega_{11}$  and  $\Omega_7 \cong \Omega_{12}$ . The argument presented in the proof of Lemma 3.2 shows that the map  $x_1 \mapsto x_1$ ,  $x_2 \mapsto x_2$ , and  $x_3 \mapsto x_3 + x_1^3$  can be extended to isomorphisms between  $\Omega_6$  and  $\Omega_{11}$  and between  $\Omega_7$  and  $\Omega_{12}$ .  $\square$

#### 4.3. Family (3).

$$\begin{aligned} K_{14} &= \langle x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_5] = x_6 \\ &\quad [x_3, x_4] = -x_6, [x_2, x_3] = x_5 \rangle; \\ K_{16} &= \langle x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_5] = x_6, \\ &\quad [x_3, x_4] = -x_6 \rangle. \end{aligned}$$

**Lemma 4.4.** *The algebras  $\Omega_{14}$  and  $\Omega_{16}$  are not isomorphic.*

*Proof.* Note that  $K_{14}$  and  $K_{16}$  are algebras with maximal class. Further,  $K_{14}/(K_{14})^5 \cong L_6$  and  $K_{16}/(K_{16})^5 \cong L_7$ . Now the argument presented in the first paragraph of the proof of Lemma 4.3 shows that  $\Omega_{14} \not\cong \Omega_{16}$ .  $\square$

#### 4.4. Family (4).

$$\begin{aligned}
K_{15} &= \langle x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5, \\
&\quad [x_1, x_5] = x_6, [x_2, x_4] = x_6 \rangle; \\
K_{17} &= \langle x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, \\
&\quad [x_2, x_3] = x_6 \rangle; \\
K_{18} &= \langle x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6 \rangle.
\end{aligned}$$

**Lemma 4.5.** *If  $\text{char } \mathbb{F} \neq 3$  then  $\Omega_{15}$ ,  $\Omega_{17}$  and  $\Omega_{18}$  are pairwise non-isomorphic; otherwise the only isomorphism among these algebras is  $\Omega_{17} \cong \Omega_{18}$ .*

*Proof.* Note that  $K_{15}/(K_{15})^5 \cong L_6$  and  $K_{17}/(K_{17})^5 \cong L_7 \cong K_{18}/(K_{18})^5$ . Applying the same method as in the proof of Lemma 4.3 yields that  $\Omega_{15} \not\cong \Omega_{17}$  and  $\Omega_{15} \not\cong \Omega_{18}$ .

Suppose that  $\text{char } \mathbb{F} \neq 3$ , and let us show that  $\Omega_{17} \not\cong \Omega_{18}$ . Assume, by contradiction, that  $f : \Omega_{17} \rightarrow \Omega_{18}$  is an isomorphism. Then  $f$  is determined by the images  $f(x_1)$  and  $f(x_2)$ . Let  $f(x_1) \equiv \alpha_1 x_1 + \alpha_2 x_2$  and  $f(x_2) \equiv \beta_1 x_1 + \beta_2 x_2$  modulo  $\Omega_{18}^2$ . Let  $\delta = \alpha_1 \beta_2 - \alpha_2 \beta_1$ . Clearly,  $\delta \neq 0$ . Note that  $f(x_3) \equiv \delta x_3 \pmod{\Omega_{18}^3}$ . Thus,  $f(x_4) \equiv \alpha_1 \delta x_4 \pmod{\Omega_{18}^4}$  and  $f(x_6) \equiv \alpha_1^3 \delta x_6 \pmod{\Omega_{18}^6}$ . Also,  $f([x_2, x_3]) \equiv \beta_1 \delta x_4 \pmod{\Omega_{18}^4}$ . We deduce that  $\beta_1 = 0$ . Now we write

$$\begin{aligned}
f(x_2) &\equiv \beta_2 x_2 + u_0 + x_1 u_1 + x_1^2 u_2 + x_1^3 u_3 \pmod{\Omega_{18}^4}, \\
f(x_3) &\equiv \delta x_3 + v_0 + x_1 v_1 + x_1^2 v_2 + x_1^3 v_3 \pmod{\Omega_{18}^4},
\end{aligned}$$

where each  $u_i$  and  $v_i$  is a linear combination of (possibly trivial) PBW monomials that do not involve  $x_1$ . Since  $f(x_3) \equiv \delta x_3 \pmod{\Omega_{18}^3}$ , we deduce that weight of  $v_0 + x_1 v_1 + x_1^2 v_2$  is at least 3. Similarly, weight of  $u_0 + x_1 u_1$  is at least 2. Note that  $[u_0, v_i] = [u_i, v_0] = 0$ . So,

$$(2) \quad 0 \equiv f([x_3, x_2]) \equiv \beta_2 \sum_{i=1}^3 [x_1^i, x_2] v_i - \delta \sum_{j=1}^2 [x_1^j, x_3] u_j \pmod{\Omega_{18}^5}.$$

Expanding out the commutators in Equation (2), we observe that  $x_1^2 x_3 v_3$  is the unique term in Equation (2) that has the highest exponent of  $x_1$ . Since  $\text{char}(\mathbb{F}) \neq 3$ , this can happen only if  $v_3 \in \Omega_{18}$ . Thus,  $x_1^2 x_3 v_3 \in \Omega_{18}^5$ . The highest exponent of  $x_1$  in Equation (2) then appears in  $x_1 x_3 v_2$  and  $x_1 x_4 u_2$ . So, these terms have to cancel out with each other. We deduce that  $u_2 \in \Omega_{18}$  and  $v_2 \in \Omega_{18}^2$ . Now Equation (2) reduces to  $\beta_2 x_3 v_1 \equiv \delta x_4 u_1 \pmod{\Omega_{18}^5}$ . This implies that  $u_1 \in \Omega_{18}^2$  and  $v_1 \in \Omega_{18}^3$ . So, if we write

$$\begin{aligned}
f(x_2) &\equiv \beta_2 x_2 + u_0 + x_1 u_1 + x_1^2 u_2 + x_1^3 u_3 + x_1^4 u_4 \pmod{\Omega_{18}^5}, \\
f(x_3) &\equiv \delta x_3 + v_0 + x_1 v_1 + x_1^2 v_2 + x_1^3 v_3 + x_1^4 v_4 \pmod{\Omega_{18}^5},
\end{aligned}$$

then each  $u_i$  and  $v_i$  is a linear combination of (possibly trivial) PBW monomials that do not involve  $x_1$ ,  $u_0$  has weight at least 2,  $v_0$  has weight at least 3, weight of

$x_1v_1 + x_1^2v_2 + x_1^3v_3 + x_1^4v_4$  is at least 4, and weight of  $x_1u_1 + x_1^2u_2$  is at least 3. Thus,

$$(3) \quad -\alpha_1^3\delta x_6 = f([x_3, x_2]) \equiv \beta_2 \sum_{i=1}^4 [x_1^i, x_2]v_i - \delta \sum_{j=1}^3 [x_1^j, x_3]u_j \pmod{\Omega_{18}^6}.$$

Arguing as in Equation (2), we deduce that Equation (3) reduces to the following:

$$-\alpha_1^3\delta x_6 \equiv \beta_2 x_3 v_1 - \delta x_4 u_j \pmod{\Omega_{18}^6}.$$

The later is possible only if  $\alpha_1 = 0$  or  $\delta = 0$ . Since we have already proved that  $\beta_1 = 0$  it follows that  $\delta = 0$  which is a contradiction.

The argument in the proof of Lemma 3.2 shows, for  $\text{char}(\mathbb{F}) = 3$ , that the map  $x_1 \mapsto x_1$ ,  $x_2 \mapsto x_2 + x_1^3$  can be extended to an isomorphism between  $\Omega_{17}$  and  $\Omega_{18}$ .  $\square$

#### 4.5. Family (5).

$$\begin{aligned} K_{23} &= \langle x_1, x_2, x_3; x_4, x_5; x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_6, [x_1, x_3] = x_5, [x_2, x_3] = x_6 \rangle; \\ K_{25} &= \langle x_1, x_2, x_3; x_4, x_5; x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_6, [x_1, x_3] = x_5 \rangle. \end{aligned}$$

**Lemma 4.6.** *The algebras  $\Omega_{23}$  and  $\Omega_{25}$  are not isomorphic.*

*Proof.* Suppose that  $\text{char } \mathbb{F} \neq 3$  and assume, by contradiction, that  $f : \Omega_{25} \rightarrow \Omega_{23}$  is an isomorphism. For  $i = 1, 2, 3$ , write  $f(x_i)$  as a linear combination of PBW monomials and assume that  $\alpha_{i_1, \dots, i_n}$ ,  $\beta_{i_1, \dots, i_n}$ , and  $\gamma_{i_1, \dots, i_n}$  are the coefficients of  $x_{i_1} \cdots x_{i_n}$  in  $f(x_1)$ ,  $f(x_2)$ , and  $f(x_3)$ , respectively. Then

$$0 = f([x_2, x_3]) \equiv (\beta_1\gamma_2 - \beta_2\gamma_1)x_4 + (\beta_1\gamma_3 - \beta_3\gamma_1)x_5 \pmod{\Omega_{23}^3},$$

and hence  $\beta_1\gamma_2 - \beta_2\gamma_3 = \beta_1\gamma_3 - \beta_3\gamma_1 = 0$ . Since  $f$  induces an isomorphism between  $\Omega_{25}/(\Omega_{25})^2$  and  $\Omega_{23}/(\Omega_{23})^2$ , we obtain that  $\beta_1 = \gamma_1 = 0$ . Note that

$$f(x_5) = f([x_1, x_3]) \equiv \alpha_1\gamma_2x_4 + \alpha_1\gamma_3x_5 \pmod{\Omega_{23}^3}.$$

Thus,  $0 = f([x_1, x_5]) \equiv \alpha_1^2\gamma_2x_6 \pmod{(\Omega_{23})^4}$ , which gives that  $\gamma_2 = 0$ . Now we calculate  $f([x_2, x_3])$  modulo  $(\Omega_{23})^4$  to show that  $\beta_2\gamma_3 = 0$ . As  $[x_4, \Omega_{23}] \leq (\Omega_{23})^4$ , we have, modulo  $(\Omega_{23})^4$ , that

$$\begin{aligned} f([x_2, x_3]) &\equiv \beta_2\gamma_3x_6 + \beta_2\gamma_{1,1}(x_6 - 2x_1x_4) - \beta_2\gamma_{1,2}x_2x_4 - \beta_2\gamma_{1,3}x_3x_4 \\ &\quad + 2(\beta_{1,1}\gamma_3 - \beta_3\gamma_{1,1})x_1x_5 + (\beta_{1,2}\gamma_3 - \beta_3\gamma_{1,2})x_2x_5 + (\beta_{1,3}\gamma_3 - \beta_3\gamma_{1,3})x_3x_5. \end{aligned}$$

This gives that  $\beta_2\gamma_3 = 0$  which implies that the images  $f(x_2)$ ,  $f(x_2)$ ,  $f(x_2)$  are linearly dependent modulo  $(\Omega_{23})^2$ , which is a contradiction.  $\square$

4.6. **Family (6).** We consider the following Lie algebras:

$$\begin{aligned} K_9 &= \langle x_1, x_2, x_3; x_4; x_5, x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_4] = x_6 \rangle; \\ K_{24}(\varepsilon) &= \langle x_1, x_2, x_3; x_4; x_5, x_6 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_3] = \varepsilon x_6, [x_2, x_4] = x_6, \\ &\quad [x_2, x_3] = x_5 \rangle. \end{aligned}$$

The family  $K_{24}(\varepsilon)$  is a parametric family of Lie algebras such that  $K_{24}(\varepsilon_1) \cong K_{24}(\varepsilon_2)$  if and only if there is a  $\nu \in \mathbb{F}$  such that  $\varepsilon_1 \nu^2 = \varepsilon_2$  (see [dG]).

**Lemma 4.7.** *The algebras  $\Omega_9$  and  $\Omega_{24}(\varepsilon)$  are not isomorphic, for all  $\varepsilon \in \mathbb{F}$ . Further,  $\Omega_{24}(\varepsilon_1) \cong \Omega_{24}(\varepsilon_2)$  if and only if  $K_{24}(\varepsilon_1) \cong K_{24}(\varepsilon_2)$ , for every  $\varepsilon_1, \varepsilon_2 \in \mathbb{F}$ .*

*Proof.* Since  $x_3 \in Z(\Omega_9) \setminus (\Omega_9)^2$ , we have that  $Z(\Omega_9) \not\leq (\Omega_9)^2$ . Let  $\varepsilon \in \mathbb{F}$ . We claim that  $Z(\Omega_{24}(\varepsilon)) \leq (\Omega_{24}(\varepsilon))^2$ . Let  $z \in Z(\Omega_{24}(\varepsilon))$  and write  $z$  as a linear combination of PBW monomials in which  $\alpha_{i_1, \dots, i_n}$  denotes the coefficient of  $x_1 \cdots x_n$ . First we compute, modulo  $(\Omega_{24}(\varepsilon))^3$ , that  $0 = [z, x_3] \equiv \alpha_1 \varepsilon x_6 + \alpha_2 x_5$ , which gives that  $\alpha_2 = 0$ . Then,

$$[z, x_2] \equiv \alpha_1 x_4 + (-\alpha_3 - \alpha_{1,1})x_5 + 2\alpha_{1,1}x_1x_4 + \alpha_{1,2}x_2x_4 + \alpha_{1,3}x_3x_4 \pmod{\Omega_{24}^4(\varepsilon)},$$

which shows that  $\alpha_1 = \alpha_3 = 0$ . Hence  $z \in (\Omega_{24}(\varepsilon))^2$ , and so  $Z(\Omega_{24}(\varepsilon)) \leq (\Omega_{24}(\varepsilon))^2$ , as claimed. This implies that  $\Omega_9 \not\cong \Omega_{24}(\varepsilon)$ .

Let us now prove the second assertion of the lemma. Without loss of generality, we assume that  $\varepsilon_2 \neq 0$ . Let  $K = K_{24}(\varepsilon_2)$  and  $\Omega = \Omega(K)$ . Suppose that  $f : \Omega_{24}(\varepsilon_1) \rightarrow \Omega$  is an algebra isomorphism. As usual, for  $i = 1, 2, 3$ , we write the images  $f(x_i)$  as linear combinations of PBW monomials, and let  $\alpha_{i_1, \dots, i_n}$ ,  $\beta_{i_1, \dots, i_n}$  and  $\gamma_{i_1, \dots, i_n}$  denote the coefficients of  $x_{i_1} \cdots x_{i_n}$  in  $f(x_1)$ ,  $f(x_2)$ , and  $f(x_3)$ , respectively. Since  $f([x_1, x_3])$ ,  $f([x_2, x_3]) \in \Omega^3$ , we deduce that  $\alpha_1 \gamma_2 - \alpha_2 \gamma_1 = 0$  and that  $\beta_1 \gamma_2 - \beta_2 \gamma_1 = 0$ . Since the images  $f(x_1)$ ,  $f(x_2)$ ,  $f(x_3)$  are linearly independent modulo  $\Omega^2$ , this gives that  $\gamma_1 = \gamma_2 = 0$ . Set  $\delta = \alpha_1 \beta_2 - \alpha_2 \beta_1$ . Since  $f(x_1)$ ,  $f(x_2)$ , and  $f(x_3)$  are linearly independent modulo  $\Omega^2$ , we have  $\delta \neq 0$ . Further,

$$f(x_4) = [f(x_1), f(x_2)] \equiv \delta[x_1, x_2] = \delta x_4 \pmod{\Omega^3}.$$

Thus,  $f([x_1, x_4]) \equiv \alpha_1 \delta x_5 + \alpha_2 \delta x_6 \pmod{\Omega^4}$ . So, modulo  $\Omega^4$ , we have,

$$\begin{aligned} f([x_2, x_3]) &\equiv (\beta_1 \gamma_4 + \beta_2 \gamma_3 + \beta_2 \gamma_{1,1})x_5 + (\varepsilon_2 \beta_1 \gamma_3 - \beta_1 \gamma_{2,2} + \beta_2 \gamma_{1,2} + \beta_2 \gamma_4)x_6 + \\ &\quad (\beta_1 \gamma_{1,2} - 2\beta_2 \gamma_{1,1})x_1x_4 + (2\beta_1 \gamma_{2,2} - \beta_2 \gamma_{1,2})x_2x_4 + (\beta_1 \gamma_{2,3} - \beta_2 \gamma_{1,3})x_3x_4. \end{aligned}$$

As  $f([x_1, x_4]) = f([x_2, x_3])$  we obtain that following equations:

$$\begin{aligned} (4) \quad & \beta_1 \gamma_4 + \beta_2 \gamma_3 + \beta_2 \gamma_{1,1} = \alpha_1 \delta; \\ (5) \quad & \varepsilon_2 \beta_1 \gamma_3 - \beta_1 \gamma_{2,2} + \beta_2 \gamma_{1,2} + \beta_2 \gamma_4 = \alpha_2 \delta; \\ (6) \quad & \beta_1 \gamma_{1,2} - 2\beta_2 \gamma_{1,1} = 0; \\ (7) \quad & 2\beta_1 \gamma_{2,2} - \beta_2 \gamma_{1,2} = 0; \\ (8) \quad & \beta_1 \gamma_{2,3} - \beta_2 \gamma_{1,3} = 0. \end{aligned}$$

Now we use the relation  $\varepsilon_1[x_2, x_4] = [x_1, x_3]$ . So, modulo  $\Omega^4$ , we have

$$\begin{aligned} f([x_2, x_4]) &\equiv \beta_1 \delta x_5 + \beta_2 \delta x_6, \\ f([x_1, x_3]) &\equiv (\alpha_1 \gamma_4 + \alpha_2 \gamma_3 + \alpha_2 \gamma_{1,1})x_5 + (\varepsilon_2 \alpha_1 \gamma_3 - \alpha_1 \gamma_{2,2} + \alpha_2 \gamma_{1,2} + \alpha_2 \gamma_4)x_6 + \\ &\quad (\alpha_1 \gamma_{1,2} - 2\alpha_2 \gamma_{1,1})x_1 x_4 + (2\alpha_1 \gamma_{2,2} - \alpha_2 \gamma_{1,2})x_2 x_4 + (\alpha_1 \gamma_{2,3} - \alpha_2 \gamma_{1,3})x_3 x_4. \end{aligned}$$

We get the following equations:

$$\begin{aligned} (9) \quad & \alpha_1 \gamma_4 + \alpha_2 \gamma_3 + \alpha_2 \gamma_{1,1} = \varepsilon_1 \beta_1 \delta; \\ (10) \quad & \varepsilon_2 \alpha_1 \gamma_3 - \alpha_1 \gamma_{2,2} + \alpha_2 \gamma_{1,2} + \alpha_2 \gamma_4 = \varepsilon_1 \beta_2 \delta; \\ (11) \quad & \alpha_1 \gamma_{1,2} - 2\alpha_2 \gamma_{1,1} = 0; \\ (12) \quad & 2\alpha_1 \gamma_{2,2} - \alpha_2 \gamma_{1,2} = 0; \\ (13) \quad & \alpha_1 \gamma_{2,3} - \alpha_2 \gamma_{1,3} = 0. \end{aligned}$$

Equations (6) and (11) imply that  $\gamma_{1,1} = \gamma_{1,2} = 0$ . Similarly  $\gamma_{2,2} = \gamma_{2,3} = \gamma_{1,3} = 0$ . Thus the system of equations above are reduced to the following:

$$\begin{aligned} (14) \quad & \beta_1 \gamma_4 + \beta_2 \gamma_3 = \alpha_1 \delta; \\ (15) \quad & \varepsilon_2 \beta_1 \gamma_3 + \beta_2 \gamma_4 = \alpha_2 \delta; \\ (16) \quad & \alpha_1 \gamma_4 + \alpha_2 \gamma_3 = \varepsilon_1 \beta_1 \delta; \\ (17) \quad & \varepsilon_2 \alpha_1 \gamma_3 + \alpha_2 \gamma_4 = \varepsilon_1 \beta_2 \delta. \end{aligned}$$

Set

$$\begin{aligned} p_1 &= (-1/2)\varepsilon_1 \beta_2 \delta^{-2} - (1/2)\varepsilon_2 \alpha_1 \gamma_3 \delta^{-3}; \\ p_2 &= (1/2)\alpha_1 \gamma_4 \delta^{-3} + \alpha_2 \gamma_3 \delta^{-3}; \\ p_3 &= \alpha_2 \delta^{-2} - (1/2)\beta_2 \gamma_4 \delta^{-3} \\ p_4 &= (-1/2)\alpha_1 \delta^{-2} - (1/2)\beta_2 \gamma_3 \delta^{-3}. \end{aligned}$$

We can check that

$$\begin{aligned} p_1(\beta_1 \gamma_4 + \beta_2 \gamma_3 - \alpha_1 \delta) + p_2(\varepsilon_2 \beta_1 \gamma_3 + \beta_2 \gamma_4 - \alpha_2 \delta) + \\ p_3(\alpha_1 \gamma_4 + \alpha_2 \gamma_3 - \varepsilon_1 \beta_1 \delta) + p_4(\varepsilon_2 \alpha_1 \gamma_3 + \alpha_2 \gamma_4 - \varepsilon_1 \beta_2 \delta) = \varepsilon_1 - \varepsilon_2 \gamma_3^2 \delta^{-2}. \end{aligned}$$

Thus, considering that  $\delta \neq 0$ , the equation  $\varepsilon_1 = \varepsilon_2 \gamma_3^2 \delta^{-2}$  follows from the equations (14)–(17). However this implies that  $K_{24}(\varepsilon_1) \cong K_{24}(\varepsilon_2)$  as claimed.  $\square$

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